

Waste Makes Haste: Bounded Time algorithms for Envy-Free Cake Cutting with Free Disposal

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Abstract

We consider the classic problem of envy-free division of a heterogeneous good ("cake") among several agents. It is known that, when the allotted pieces must be connected, the problem cannot be solved by a finite algorithm for 3 or more agents. Even when the pieces may be disconnected, no bounded-time algorithm is known for 5 or more agents. The impossibility result, however, assumes that the entire cake must be allocated. In this paper we replace the entire-allocation requirement with a weaker *partial-proportionality* requirement: the piece given to each agent must be worth for it at least a certain positive fraction of the entire cake value. We prove that this version of the problem is solvable in bounded time even when the pieces must be connected. We present bounded-time envy-free cake-cutting algorithms for: (1) giving each of n agents a connected piece with a positive value; (2) giving each of 3 agents a connected piece worth at least $1/3$; (3) giving each of 4 agents a connected piece worth at least $1/7$; (4) giving each of 4 agents a disconnected piece worth at least $1/4$; (5) giving each of n agents a disconnected piece worth at least $(1 - \epsilon)/n$ for any positive ϵ .

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1 Introduction

Fair cake-cutting is an active field of research with applications in mathematics, economics, and recently also in AI (Procaccia, 2015). The basic setting considers a heterogeneous good, usually described as a one-dimensional interval, that must be divided among several agents. The different agents may have different preferences over the possible pieces of the good. The goal is to divide the good among the agents in a way that is deemed “fair”. Fairness can be defined in several ways, of which *proportionality* and *envy-freeness* are the most commonly used.

Proportionality means that each agent gets at least its “fair-share” of the good, i.e. with n agents, the piece allotted to each agent is worth for him at least $1/n$ of the value of the entire good. *Envy-freeness* means that every agent believes that its piece is weakly better than any other piece - no agent would prefer to get a piece allotted to another agent. An additional requirement in cake-cutting, particularly relevant when the divided resource is land, is *connectivity* - each agent must be given a single contiguous piece.

Proportional division is a relatively easy task, and the initial work of Steinhaus (1948) already provided an algorithm for n agents with connected pieces. The algorithm works in a *query model*: it asks the agents queries such as what is the value of this piece for you?” or ”what piece is worth $1/n$ of the cake for you?” and proceeds according to their replies. The number of queries required by Steinhaus’ algorithm is polynomial in the number of agents.

Envy-free division, on the other hand, turns out to be much more challenging. With connected pieces, the only algorithm for envy-free division is an infinite one; that is, it may require an infinite number of queries to reach an envy-free division (Su, 1999). Indeed, Stromquist (2008) proved that this is necessarily so; any algorithm for computing an envy-free division with connected pieces must require an infinite number of queries on some inputs. This is so even when there are only 3 agents! With disconnected pieces, three finite algorithms are known. However, their run-time is not bounded - they might require an arbitrarily large number of queries on some inputs.

A closer examination of these discouraging results reveals that they critically rely on the assumption that *the entire cake must be divided*. In many practical situations, it may be possible to leave some parts of the cake undivided, a possibility termed *free disposal*. If, for example, your children spend too much time quarreling over the single cherry on top of the cake, one practical solution is to eat that cherry yourself and divide only the rest of the cake. As another example, when dividing land it is usually possible (and sometimes even desirable) to leave some parts of the land unallocated, so that they can be used freely by the public. The question of interest in this paper is thus:

If free disposal is allowed, can an envy-free allocation be computed using a bounded number of queries?

This question, however, turns out to have a trivial, but uninteresting, answer; It is always possible to give nothing to all agents, which is an envy-free allocation.

Thus, the interesting question is whether it is possible to devise a bounded-time algorithm for envy-free division in which each agent gets a *strictly positive* value.

1.1 Results

Our first algorithm provides an affirmative answer to this question (Section 4):

Theorem 1. *There is a bounded-time algorithm that divides a cake among n agents in an envy-free way, giving each agent a connected piece with a positive value, assuming free disposal.*

Having established that bounded-time algorithms indeed exist, we next consider the *quality* of the solution they offer. We measure the quality of an algorithm by the value guarantee it gives to each agent. We say that an allocation is **EnvyFree** $[n, M]$ if it is both envy-free and gives each of n agents at least $1/M$ of the total cake value (a formal definition is given in Section 2). The algorithm of Theorem 1 finds **EnvyFree** $[n, 2^{n-1}]$ allocations: in the worst case some agents might get as little as $1/2^{n-1}$ of the total cake value. Ideally, we would like an **EnvyFree** $[n, n]$ allocation, which is both envy-free and proportional and thus satisfies the two most common fairness criteria. Note that such an allocation guarantees each agent at least $1/n$ of the total cake value, which is the largest fraction that can be guaranteed (for example, if all agents have identical valuations, it is impossible to give all of them more than $1/n$).

Our next result shows that the ideal goal of an envy-free and proportional division with connected pieces can be attained in bounded time for three agents (Section 5):

Theorem 2. *There is a bounded-time algorithm that finds an **EnvyFree** $[3, 3]$ allocation with connected pieces, assuming free disposal.*

For 4 agents, we do not have an envy-free-proportional algorithm with connected pieces. Our best result so far guarantees each agent $1/7$ of the total cake value (Section 6):

Theorem 3. *There is a bounded-time algorithm that finds an **EnvyFree** $[4, 7]$ allocation with connected pieces, assuming free disposal.*

While this algorithm does not solve the envy-free-proportional problem with connected pieces, it is useful as a building block for cake-cutting with disconnected pieces, as we describe next.

Given a pre-specified agent i (which we call the "VIP"), we say that an allocation is **EnvyFreeVIP** $[n, M]$ if it is both envy-free and gives the VIP a piece worth at least $1/M$ of the total cake value. We prove the following reductions (Sections 7-8):

Lemma 1. (a) *If we can find **EnvyFreeVIP** $[n, M]$ allocations using T queries then we can find **EnvyFree** $[n, M]$ allocations using $n \cdot T$ queries.*

(b) *If we can find **EnvyFreeVIP** $[n, M]$ allocations using T queries, then for every $\epsilon > 0$ we can find **EnvyFreeVIP** $[n, \frac{n}{1-\epsilon}]$ allocations using $\frac{M \ln(1/\epsilon)}{n} \cdot T$ queries.*

The allocations found by the algorithm of Theorem 3 are not only $\text{EnvyFree}[4, 7]$ but also $\text{EnvyFreeVIP}[4, 4]$. This allows us to provide an alternative proof to a result of Saberi and Wang (2009):

Theorem 4. *There is a bounded-time algorithm that finds $\text{EnvyFree}[4, 4]$ allocations with disconnected pieces, assuming free disposal.*

More importantly, the algorithm used in Theorem 1 finds $\text{EnvyFreeVIP}[n, 2^{n-2} + 1]$ allocations using $O(2^n)$ queries. This gives:

Theorem 5. *For every $\epsilon > 0$, there is an algorithm that finds $\text{EnvyFree}[n, \frac{n}{1-\epsilon}]$ allocations using $O(4^n \ln(1/\epsilon))$ queries, assuming free disposal.*

This means that we can find, in bounded time, an envy-free division which is as close as we want to a proportional division (each agent receives at least $(1 - \epsilon)/n$ of the total cake value). The number of queries is linear in the binary representation of the approximation factor (ϵ).

All our algorithms are very simple and use only a single type of query: the Equalize query (defined in Section 3). The hard work is done in the correctness proofs.

While we worked on envy-free cake-cutting with free disposal, Aziz and Mackenzie (2015) published a preprint of a bounded-time envy-free cake-cutting algorithm with no disposal, for 4 agents. We found out that the algorithm of our Theorem 3 can be combined with some of their ideas to yield a simpler algorithm for the same task. We believe that this simplification can help future researchers find more advanced cake-cutting algorithms. This contribution is given in Appendix B.

Table 1 summarizes our results and compares them to some related work surveyed in the next subsection.

1.2 Related work

1.2.1 Cake-cutting problems

Cake-divisions are commonly modeled in one of two ways: in the *connected* model, the algorithm must give each agent a single contiguous piece; in the *unrestricted* model, the algorithm may give each agent several disconnected pieces.

Proportional division is well understood from a computational perspective. The algorithm of Steinhaus (1948) generates a proportional division with connected pieces in $O(n^2)$ queries, and an improved algorithm by Even and Paz (1984) requires only $O(n \log n)$ queries. Later results proved that this runtime is asymptotically optimal even if disconnected pieces are allowed (Woeginger and Sgall, 2007; Edmonds and Pruhs, 2011).

Envy-free division is a much harder task, even when only 3 agents are involved. The first envy-free division algorithm for 3 agents with connected pieces was published by Stromquist (1980). This algorithm is not discrete - it requires

Name	Pieces	Agents	Valuations	#Queries	Envy	Prop.
Stromquist (1980)	Con.	3	General	Infinite	0	$1/3^*$
Su (1999)	Con.	n	General	Infinite	0	$1/n^*$
Deng et al. (2012)	Con.	n	Lipschitz	$O((1/\epsilon)^{n-2})$	ϵ	$1/n - \epsilon$
Brânzei (2015)	Con.	n	Polynomials	$O(n^2 \cdot d)$	0	$1/n^*$
Selfridge-Conway	Dis.	3	General	Constant	0	$1/3^*$
Saberi and Wang (2009)	Dis.	4	General	Constant	0	$1/4^*$
Aziz and Mackenzie (2015) (see also our Appendix B)	Dis.	4	General	Constant	0	$1/4^*$
Reentrant-diminisher (Brams and Taylor, 1996)	Dis.	n	General	$O(n^2/\epsilon)$	ϵ	$1/n^*$
Brams and Taylor (1995) Robertson and Webb (1998) Pikhurko (2000)	Dis.	n	General	Unbounded	0	$1/n^*$
Kurokawa et al. (2013)	Dis.	n	Piecewise-lin.	$O(n^6 k \ln k)$	0	$1/n^*$
Section 4	Con.	n	General	$O(2^n)$	0	$1/2^{n-1}$
Section 5	Con.	3	General	Constant	0	$1/3^*$
Section 6	Con.	4	General	Constant	0	$1/7$
Section 7	Dis.	4	General	Constant	0	$1/4^*$
Section 8	Dis.	n	General	$O(4^n \ln(1/\epsilon))$	0	$1/n - \epsilon/n$

Table 1: Previous envy-free algorithms (top) and our algorithms (bottom).

Legend:

- Pieces column: whether the pieces are **C**onected or **D**isconnected.
- Queries column: d and k are parameters of the valuation functions (maximum degree of polynomials and number of pieces, respectively).
- Envy column: ϵ is an additive approximation constant (every agent values other pieces at most ϵ more than its own piece).
- Prop column: the expression is the proportion of the total cake value that is guaranteed to all agents. $1/n^*$ implies that the division is proportional.

the agents to simultaneously hold knives over the cake and move them in a continuous manner. This means that this algorithm cannot be accurately executed by a computer in finite time. A discrete and finite algorithm for envy-free division for 3 agents was constructed by Selfridge and Conway Brams and Taylor (1996), but it generates partitions with disconnected pieces.

Finding an envy-free division among four or more agents was a long-standing open problem. It was solved only in the 1990's, both for connected and disconnected pieces. Su (1999) presented an algorithm, attributed to Forest Simmons, for envy-free division with connected pieces, but it is not finite - it converges to an envy-free division after a possibly infinite number of queries. Brams and Taylor (1995), Robertson and Webb (1998) and Pikhurko (2000) presented three different algorithms for envy-free division with disconnected pieces; while these algorithms are guaranteed to terminate in finite time, their run-time is not a bounded function of n . The most recent breakthrough is due to Aziz and Mackenzie (2015), who published a pre-print of the first bounded-time algorithm for 4 agents (with disconnected pieces).

Two important hardness results were proved in the 2000's. Stromquist (2008) proved that an envy-free division with connected pieces cannot be found by any finite algorithm, whether bounded or unbounded. This shows that the connectivity requirement makes the envy-free division problem strictly more difficult. Shortly afterward, Procaccia (2009) proved an $\Omega(n^2)$ lower bound on the query complexity of any envy-free division algorithm, even with disconnected pieces. This proved that the problem of envy-free division is strictly more difficult than the problem of proportional division.

1.2.2 Approximations

Cake-cutters have tried to cope with the difficulty of envy-free division in several ways.

One way is to relax the envy-freeness criterion and allow a small amount of envy. Brams and Taylor (1996) describe a re-entrant variant of Steinhaus' algorithm which produces a division with disconnected pieces in which the envy of every agent is at most an additive constant ϵ (for every agent, the value of its piece plus ϵ is at least the value of any other piece). The number of queries is polynomial in n and linear in $(1/\epsilon)$. Deng et al. (2012) present a similar approximation with connected pieces; here the number of queries is exponential in n and polynomial in $(1/\epsilon)$. In contrast to these results, our algorithms guarantee full envy-freeness. Our algorithm for disconnected piece also guarantees an additive approximation to proportionality. The number of queries in our approximation is exponential in n but *logarithmic* in $(1/\epsilon)$ (in other words, it is linear in the binary representation of the approximation constant).

A second way is to restrict the value function of the agents. Kurokawa et al. (2013) require the value functions to be piecewise-linear and find an envy-free division with disconnected pieces in time polynomial in the size of the representation of the value functions. Deng et al. (2012) require the value functions to be Lipschitz-continuous and find an approximately-envy-free division with

connected pieces. Brânzei (2015) requires the value functions to be polynomials of bounded degree and finds an envy-free division with connected pieces in time polynomial in the maximum degree. In contrast to these results, our algorithms apply to arbitrary non-atomic value functions, and their runtime guarantee is a function of only the number of agents but not the peculiarities of their valuation functions.

1.2.3 Free disposal

The free disposal assumption was introduced into envy-free cake-cutting by Saberi and Wang (2009). They used it only for 4 agents and disconnected pieces.

Later, free disposal has also been studied by Arzi et al. (2011). They proved that discarding some parts of the cake may allow us to achieve an envy-free division with an improved social welfare (i.e. the sum of the utilities of the agents is larger than in the no-free-disposal case). They call this phenomenon the *dumping paradox*. Our paper demonstrate a different kind of a dumping paradox - we show that dumping some parts of the cake can be beneficial not only from an economic perspective but also from a computational perspective.

A third scenario in which free disposal is required is when the pieces must have a pre-specified geometric shape, such as a square (Segal-Halevi et al., 2015a).

There is some related work concerning allocation of indivisible goods where the same idea of not allocating all the objects is used to get better fairness results (Brams et al., 2013; Aziz, 2014).¹

Partial proportionality was introduced by Edmonds and Pruhs (2006); Edmonds et al. (2008), who used it, like us, to reduce the query complexity. They presented an algorithm for finding a partially-proportional division with a query complexity of $O(n)$, which is better than the optimum of $O(n \log n)$ required for finding a fully-proportional division.

1.2.4 Computational models

The most prominent computational model for discrete cake-cutting is the mark-eval model of Robertson and Webb (1998). A different model, the cut-choose model, was recently suggested by Brânzei et al. (2013). Our algorithms use a single primitive query - Equalize (defined in Section 3). An Equalize query can be implemented by a bounded number of mark-eval queries or cut-choose queries. Hence, our algorithms are bounded in both these models.

1.3 Paper structure

The model is formally defined in Section 2. The main tools used in our division algorithms, the *preference graph* and the *Equalize query*, are introduced in Section 3.

¹We thank an anonymous reviewer for referring us to these papers.

Sections 4-8 are devoted to proving the theorems. For every $k \in \{1, \dots, 5\}$, Theorem k is proved in Algorithm k and Section $k + 3$.

A detailed, computer-generated proof of the four-agents algorithm of Section 6 is given in Appendix A. An application of that algorithm to envy-free division with no disposal is presented in Appendix B.

2 Model and Notation

The cake is assumed to be the unit interval $[0, 1]$.

There are n agents, denoted by A_1, A_2, \dots, A_n . When the number of agents is small, they are denoted instead by A, B, C, ... or by Alice, Bob, Carl...

An *allocation* of a cake is an n -tuple of pairwise-disjoint subsets of the cake: $X_1 \cup \dots \cup X_n \subseteq [0, 1]$. When *connected* pieces are required, each piece X_i must be an interval; when *disconnected* pieces are allowed, each piece X_i may be a finite union of intervals.

Each agent A_i has a preference relation \succeq_i that is represented by a non-negative value measure V_i on the pieces. The term "measure" implies that it is additive - the value of a piece is equal to the sum of the values of its parts. All value measures are absolutely continuous with respect to length. This implies that all singular points have a value of 0 to all agents, i.e. there are no valuable "atoms" which cannot be divided. The value measures are normalized such that $\forall i : V_i([0, 1]) = 1$. All these assumptions are standard in the cake-cutting literature.

An allocation X is called *envy-free* if each agent values his allocated piece at least as much as every other allocated piece:

$$\forall i, j \in \{1, \dots, n\} : V_i(X_i) \geq V_i(X_j) \quad (\text{Equivalently: } X_i \succeq_i X_j)$$

We say that an allocation X has a *proportionality* of $1/M$ if it allocates each agent a fraction of at least $1/M$ of the total cake value:

$$\forall i \in \{1, \dots, n\} : V_i(X_i) \geq 1/M$$

An allocation with a proportionality of $1/n$ is usually called a *proportional* allocation.

An allocation X is called **EnvyFree** $[n, M]$ if it is both envy-free and has a proportionality of $1/M$. Note that every envy-free allocation of the entire cake is EnvyFree $[n, n]$,² but this is not necessarily true when some cake remains unallocated.

Given a pre-specified agent A_i , an allocation X is called **EnvyFreeVIP** $[n, M]$ with VIP A_i if it is both envy-free and:

$$V_i(X_i) \geq 1/M$$

²An envy-free allocation gives each agent a piece which is best (for that agent) of n pieces. By the pigeonhole principle, the best of n is worth at least $1/n$. Hence, an envy-free allocation of an entire cake has a proportionality of $1/n$.

3 Tools

Our algorithms are described in a bottom-up approach. We first present basic tools that perform well-defined tasks, then combine these tools to get a full algorithm. We believe that the bottom-up approach may be beneficial to future cake-cutters, that may use our tools to develop improved algorithms.

3.1 The preference graph

At any time during the execution of an algorithm, there is a certain number of pieces on the table, which together comprise the entire cake. The *preference graph* is a bipartite graph, in which the nodes in one side represent the n agents and the nodes in the other side represent the pieces. The pieces are denoted as numbers with a hat, e.g. $\hat{1}$, $\hat{2}$, etc. There is an edge from an agent A_k to a piece \hat{i} if A_k *prefers* \hat{i} , i.e., for every piece \hat{j} : $\hat{i} \succeq_k \hat{j}$. Note that an agent can "prefer" two or more pieces. This means that the agent is indifferent between these pieces but values any of them more than any other piece. Here are two possible preference graphs for three agents:



Both graphs may be the result of Alice cutting the cake to 3 pieces which are equal in her eyes. In the left graph, Bob and Carl each prefer a different piece; in the right graph, they prefer the same piece ($\hat{3}$).

A *saturated matching* in a bipartite graph is a subset of the edges, in which each agent-node has a single neighbor and each piece-node has at most a single neighbor. A saturating matching in the preference graph corresponds to an envy-free allocation of the cake, since every agent is allocated a preferred piece.

A well-known tool for proving the existence of saturated matchings in bipartite graphs is *Hall's marriage theorem*. This theorem, applied to our setting, implies the following lemma:

Lemma 1. *An envy-free allocation exists, if and only if for every $k = 1, \dots, n$, every group of k agents jointly prefers at least k pieces.*

In a preference graph, Hall's condition is always satisfied for groups of $k = 1$ agents since every agent has at least one preferred piece.

In the left graph above, Hall's condition is also satisfied for every group of 2 or 3 agents; this means that an envy-free allocation exists. Indeed, the allocation A- $\hat{1}$, B- $\hat{2}$ and C- $\hat{3}$ is envy-free.

In the right graph above, Hall's condition is violated by the group $\{B, C\}$. This means that an envy-free allocation cannot be attained using the existing pieces. In this case, the graph should be *transformed* in order to create a graph that meets Hall's condition. The main query we use to transform the preference graph is the *Equalize* query, which is described in the next subsection.

3.2 The Equalize query

Given an integer $k \in \{2, \dots, n\}$, the query $\text{Equalize}(k)$ asks an agent to mark zero or more pieces such that, if the pieces are cut according to these marks, that agent will have at least k best pieces. For example, in right graph above, an $\text{Equalize}(2)$ query to Bob implies the following question: "where would you cut piece $\hat{3}$, your currently favorite piece, such that you will have two equally-best pieces?".

Suppose Bob's second-best piece is $\hat{2}$. Bob can answer the $\text{Equalize}(2)$ question in one of two ways:

1. If $V_B(\hat{3}) \geq 2V_B(\hat{2})$, then $\hat{3}$ should be cut to two pieces of equal value, which is $V_B(\hat{3})/2$.
2. Otherwise, $\hat{3}$ should be cut to two unequal pieces - one having a value of $V_B(\hat{2})$ and the other having a smaller value $V_B(\hat{3}) - V_B(\hat{2})$.

A third option is that $V_B(\hat{3}) = V_B(\hat{2})$. In this case, no cutting is needed since Bob already has two pieces of equal value and better than the third piece. Here and in the rest of the paper, we ignore such fortunate coincidences. This does not lose generality, since it only makes it harder to find an envy-free division - it decreases the number of edges in the preference graph and makes it harder to find a saturated matching.

Formally, we make the following assumption about the preference graph:

Assumption 1. *After an agent A cuts a piece \hat{i} , if an agent $B \neq A$ prefers a piece $\hat{j} \neq \hat{i}$, then B does not prefer \hat{i} .*

Assumption 1 makes the descriptions of division algorithms simpler since it reduces the number of cases that need to be handled. We now prove that this simplicity does not lose generality.

Lemma 2. *If there is an algorithm P that finds $\text{EnvyFree}[n, M]$ allocations when Assumption 1 is satisfied, then there is an algorithm P' that finds $\text{EnvyFree}[n, M]$ allocations even when Assumption 1 is violated.*

Proof. We assume that the existing algorithm P is given as a service, that accepts the current preference graph and replies with the next *Equalize* query to issue.

The new algorithm P' uses this service to simulate P . It issues the *Equalize* query sent by P to the agents, collects the agents' replies, makes the required cuts and updates the preference graph. If the new preference graph violates

assumption 1, so that e.g. after agent A cut \hat{i} agent B prefers both $\hat{j} \neq \hat{i}$ and \hat{i} , then P' removes the edge $B \rightarrow \hat{i}$ and sends to P the reduced graph. The modified graph is also a possible outcome of $\text{Equalize}(k)$ and it satisfies assumption 1, so P must know how to handle it. Hence, eventually the simulation is terminated and the reduced preference graph has a saturated matching which corresponds to an $\text{EnvyFree}[n, M]$ allocation. The same matching is also a saturated matching on the real preference graph, since the real preference graph contains (at least) all the edges of the reduced graph. Hence, P' returns an $\text{EnvyFree}[n, M]$ allocation. \square

Assumption 1 has several simple corollaries which are implicitly used below. For every set X of pieces, define the *last cutter* of X to be the last agent who made a cut on any piece from the set X .

- If an agent A prefers a set X of pieces with $|X| \geq 2$, then A is the last cutter of X .
- Each two agent-nodes in the preference graph have at most one neighbor at common (there is at most one piece that both agents prefer).
- If an agent cuts the cake to several equal pieces, then every other agent prefers exactly one piece (as in the graphs above).

We now return to the three-agent example. If the algorithm implements Bob's suggested cuts, the preference graph is transformed. If Bob gave an answer of type 1, it is transformed as in the left graph below; if Bob gave an answer of type 2, it is transformed as in the right graph:



Note that in both cases, the edge $A-\hat{3}$ is gone, because piece $\hat{3}$ has been trimmed by Bob so its value for Alice is probably smaller. By Assumption 1, we ignore the fortunate coincidence in which Bob trimmed a part which happens to be worthless for Alice. The edges $A-\hat{1}$ and $A-\hat{2}$ remain, because these two pieces were not touched by Bob (Alice is still the last cutter of these pieces).

The dotted edges emanating from C imply that we do not know which piece is preferred by Carl after the cuts, since his previously-best piece - $\hat{3}$ - has been trimmed. On the other hand, we know that Bob now prefers two pieces - one of them is the trimmed $\hat{3}$ and the other is another piece, which is either his previously-second-best piece $\hat{2}$ or the new piece $\hat{4}$.

Even though we don't know which piece is now preferred by Carl, we can be sure that a saturated matching exists. This follows from the following lemma:

Lemma 3. *Suppose the n agents are divided to $n - 1$ agents whose preferences are known and one agent whose preferences are unknown. If for every $k = 1, \dots, n - 1$, every group of k known agents jointly prefers at least $k + 1$ pieces, then an envy-free allocation exists.*

Proof. Suppose every group of k known agents jointly prefers at least $k + 1$ pieces. Then, for any possible preference of the unknown agent, every group of $k + 1$ agents jointly prefers at least $k + 1$ pieces. Additionally, every group of 1 agent always prefers at least 1 piece. Hence, by Lemma 1 an envy-free allocation exists. \square

In the graphs above, there are two known agents - A and B, and one unknown agent - C. Each of the known agents prefers two pieces, and the two of them together prefer 3 or 4 pieces. Hence, whatever C's preferences are, a saturated matching exists and an envy-free allocation can be found.

3.3 Example: an algorithm for 3 agents

By now, we have described an envy-free division algorithm for three agents. The algorithm can be succinctly summarized by the following two statements:

Alice: Equalize(3)
Bob: Equalize(2)

In words: the algorithm asks Alice to cut the cake to 3 equal pieces in her eyes, then asks Bob to cut one of these pieces in order to make 2 equally-best pieces in his eyes. The outcome always looks like one of the preference graphs above, which means that a saturated matching exists and each agent can be allocated a best piece.

The last step of the algorithm is to actually find the matching and implement the corresponding envy-free allocation. To do this in our case, it is sufficient to ask Carl to pick his best piece, then ask Bob to pick one of his best pieces (which must be the piece that he trimmed, if it is still available), then ask Alice to pick a remaining piece. In the rest of this paper, we suppress this last step from the description of our algorithms. Since a maximum matching in a bipartite graph can always be found in polynomial time, it is sufficient to prove that the algorithm guarantees that a saturated matching exists.

3.4 The Envy-Free-Proportionality Lemma

We now calculate the proportionality of the resulting allocation - the value guarantee per agent. This is based on a general lemma which we call the EFP (Envy-Free-Proportionality) lemma:

Lemma 4. *(EFP Lemma) If a cake is partitioned to a set of $M \geq n$ pieces and each agent receives a single preferred piece from that set, then the allocation is $\text{EnvyFree}[n, M]$.*

Proof. Envy-freeness is obvious since each agent receives one of his best pieces. Proportionality is a result of the fact that the value functions of the agents are measures, so they are additive. The sum of the values of all pieces is the value of the entire cake. Hence, by the pigeonhole principle, the value of any best piece is at least $1/M$ of the total cake value. \square

Going back to our three-agents algorithm, we see that the algorithm partitions the cake to $M = 4$ pieces. Hence, by the EFP lemma, it generates an $\text{EnvyFree}[3, 4]$ allocation. This is only a warm-up algorithm; the algorithm of Section 5 generates $\text{EnvyFree}[3, 3]$ allocations, which are optimal (in terms of proportionality) for 3 agents.

Before continuing with more advanced division algorithms, we need two more tools: an assumption and a lemma.

3.5 The new-pieces assumption

In all algorithms studied in this paper, the first query is an Equalize query, asking one of the agents to cut n or more equal pieces (like the "Alice:Equalize(3)" in Subsection 3.3). In the rest of this paper, we make the following additional assumption on the preference graph starting with the second query:

Assumption 2. *In any query after the first query, when a new piece is created by a cut, it is not the preferred piece of any agent.*

So in the example of Subsection 3.2, we assume that after "Bob:Equalize(2)" the preference graph looks like the rightmost graph. This assumption does not lose generality because, from Hall's perspective, it only makes it harder to find a saturated matching - it concentrates the same number of edges over a smaller number of piece nodes. Formally:

Lemma 5. *If there is an algorithm P that finds $\text{EnvyFree}[n, M]$ allocations when Assumption 2 is satisfied, then there is an algorithm P' that finds $\text{EnvyFree}[n, M]$ allocations even when Assumption 2 is violated.*

Proof. Similarly to Lemma 2, P' simulates P by reading the next Equalize query, issuing it to the agents and updating the preference graph.

Suppose the m -th query in P ($m \geq 2$) is $\text{Equalize}(k)$. This query creates $k - 1$ new pieces. The algorithm P' defines an injection f_m , which maps every new piece \hat{j} to a unique original piece $f_m(\hat{j})$. By "original piece" we mean one of the pieces generated by the first Equalize query. There are at least n original pieces and $k \leq n$, so an injection f_m always exists.

P' then constructs a reduced preference graph, by converting any edge $A_i \rightarrow \hat{j}$ (for every agent A_i and new piece \hat{j}) to an edge $A_i \rightarrow f_m(\hat{j})$. P' then sends the reduced graph to P . The reduced graph corresponds to a possible outcome of $\text{Equalize}(k)$ and it satisfies Assumption 2, so P must know how to handle it. Hence, eventually the simulation is terminated and the reduced preference graph has a saturated matching. This matching corresponds to an $\text{EnvyFree}[n, M]$

allocation in which each A_i receives a piece X_i , which is one of the original pieces.

For every original piece X_i , define the set $Y_i = \{X_i\} \cup \{\hat{j} | f_m(\hat{j}) = X_i, m \geq 2\}$. This is the set of all pieces that were mapped to X_i at some point during the simulation of P . The sets Y_i , for $i = 1, \dots, n$, are pairwise-disjoint.

In the reduced graph, there is a preference edge $A_i \rightarrow X_i$. By construction, this preference edge comes from some edge in the real graph, $A_i \rightarrow \hat{j}$ where $\hat{j} \in Y_i$. Hence, for every A_i , the set Y_i contains some piece which is a best piece for A_i .

As a final step, P' asks each A_i to select a best piece from the set Y_i . The total number of pieces (M) is unchanged. Hence, the resulting allocation is $\text{EnvyFree}[n, M]$. \square

Hence, the new pieces are omitted from the preference graphs; only their total number is kept in mind for the sake of calculating the proportionality.

3.6 The unknown-agent lemma

Lemma 6. *Suppose the n agents are divided to $n - 1$ agents whose preferences are known and one agent whose preferences are unknown. If every known agent prefers at least 2 pieces, then an envy-free allocation exists.*

Proof. By Lemma 3, it is sufficient to prove that every k known agents jointly prefer at least $k + 1$ pieces. The proof is by induction on k . The base $k = 1$ is given. Suppose the lemma is true for all groups of less than k known agents. Consider a group of k known agents A_1, \dots, A_k . Each of these agents prefers at least two pieces. By Assumption 1, each of these agents was the last one to cut at least one of his two preferred pieces. Suppose that the last agent to cut one of his preferred pieces was A_1 . Suppose that A_1 prefers pieces $\hat{1}, \hat{2}$ and that he was the last agent to cut $\hat{1}$.

By Assumption 1 again, any other agent that prefers $\hat{1}$ does not prefer any other piece. This means that any other known agent does not prefer $\hat{1}$. By the induction assumption, agents A_2, \dots, A_k jointly prefer k pieces, which must be different than $\hat{1}$. With $\hat{1}$, agents A_1, \dots, A_k jointly prefer $k + 1$ pieces. \square

4 Connected pieces and n Agents

Generalizing the 3-agent algorithm from the Section 3 to n agents requires the following building blocks: an $\text{Equalize}(k)$ query for arbitrary k , and a generalized version of Lemma 6. We now describe each of these generalizations in turn.

Answering an $\text{Equalize}(k)$ query for $k \geq 3$ is a non-trivial task. There are many different options. For example, a reply to $\text{Equalize}(3)$ can have one of the following forms:

1. Cutting the best piece to three equal pieces, which are all better than the previously second-best piece; or -

2. Trimming the best piece such that it is twice as valuable as the second-best piece, then cutting the result to two halves; or -
3. Trimming both the best and the second-best pieces, such that the trimmed pieces are equal to the third-best piece.

Naturally, the number of options grows as k becomes larger.

Fortunately, $\text{Equalize}(k)$ can be answered in bounded time. We prove this by reducing $\text{Equalize}(k)$ to the following problem:

EnvyFreeStickDivision $[m, k]$: Given m sticks of different lengths, make a minimal number of cuts such that there are at least k pieces with equal lengths and no other piece is longer.

Reitzig and Wild (2015) have recently presented an algorithm that solves the envy-free stick-division problem in time $O(m)$. The algorithm works as follows. First, based on the lengths of the m sticks, it calculates an "optimal length", l^* . This is defined as the largest l such that it is possible to cut at least k pieces of length l . Then, it cuts l^* -sized pieces off of any stick longer than l^* until all sticks have length at most l^* . We use their algorithm to prove the following lemma.

Lemma 7. *It is possible to calculate an agent's answer to an $\text{Equalize}(k)$ query using a bounded number of mark and eval queries.*

Proof. Suppose there are currently m pieces on the table, X_1, \dots, X_m . Use m eval queries to find the agent's valuations to these m pieces, $V_i(X_1), \dots, V_i(X_m)$. Create m sticks, such that the length of stick j is $V_i(X_j)$. Use the algorithm of Reitzig and Wild (2015) for $\text{EnvyFreeStickDivision}[m, k]$ to find the optimal length l^* . Using k mark queries, cut k pieces that the agent values as exactly l^* . By definition of $\text{EnvyFreeStickDivision}$, the values of all other pieces are at most l^* , so this is indeed a solution to $\text{Equalize}(k)$. \square

The next tool we need is a generalization of Lemma 6.

Lemma 8. *Suppose the n agents are divided to $n - u$ agents whose preferences are known and u agents whose preferences are unknown. If every known agent prefers at least $1 + 2^{u-1}$ pieces, then an envy-free allocation can be attained with a bounded number of queries.*

Proof. The proof is by induction on u . The base $u = 1$ is Lemma 6. Assume the claim is true for u ; we have to prove it for $u + 1$ unknown agents.

Suppose the known agents are A_1, \dots, A_{n-u-1} and the unknown agents are A_{n-u}, \dots, A_n . Suppose that every known agent prefers at least $1 + 2^u$ pieces. We have to prove that an envy-free allocation can be attained with a bounded number of queries.

Ask agent A_{n-u} to $\text{Equalize}(1 + 2^{u-1})$. This requires it to trim at most 2^{u-1} pieces and guarantees that it prefers $1 + 2^{u-1}$ pieces. Every known agent still prefers at least $(1 + 2^u) - 2^{u-1} = 1 + 2^{u-1}$ pieces. Hence, by adding A_{n-u}

Algorithm 1 Finding $\text{EnvyFree}[n, 2^{n-1}]$ allocations with connected pieces.

For $u = n - 1$ to 1:

A_{n-u} : $\text{Equalize}(2^{u-1} + 1)$.

to the set of known agents, the situation becomes exactly as in the induction assumption: there are u unknown agents each of whom prefers $1 + 2^{u-1}$ pieces. Hence, by the induction assumption an envy-free allocation can be attained with a bounded number of queries. \square

The proof of Lemma 8 above immediately translates to a division algorithm (Algorithm 1). Initially, all agents are unknown. A_1 is asked to $\text{Equalize}(2^{n-2} + 1)$ and becomes a known agent. Each step, another agent is asked to Equalize and becomes a known agent. Finally, A_{n-1} is asked to $\text{Equalize}(2)$; then, each of the first $n - 1$ agents prefers at least 2 pieces, and by Lemma 6 a saturated matching exists.³

Initially there is a single piece (the entire cake). Each Equalize action adds 2^{u-1} new pieces. Hence the total number of pieces after the last cut is:

$$1 + \sum_{i=0}^{n-2} 2^i = 2^{n-1}$$

By the EFP Lemma, the division is envy-free, and each agent receives a single connected pieces with a value of at least:

$$\frac{1}{2^{n-1}}$$

which proves our Theorem 1.

Remark 1. *The algorithm presented above is similar to an algorithm mentioned by Brams and Taylor (1996) (chapter 7, page 135) as a sub-routine of their unbounded algorithm for envy-free cake-cutting with disconnected pieces. However, their sub-routine does not use the generalized Equalize query and hence does not guarantee any positive proportionality with connected pieces.*

5 Connected pieces and 3 Agents

Our goal in this and the next section is to improve the proportionality from the exponential figure guaranteed by the algorithm of Section 4. In this section we focus on the case of 3 agents. We first note that any algorithm starting with a pre-specified agent cutting 3 equal pieces cannot guarantee a proportionality of more than $1/4$ with connected pieces (since the values of these pieces in the eyes of the other two agents might be $1/2$, $1/4$ and $1/4$). However, when the cutting agent can be selected according to preferences, the optimal proportionality - $1/3$ - is attainable. This can be done by Algorithm 2.

³The matching can be implemented by letting the agents pick pieces in reverse order, from n to 1.

Algorithm 2 Finding $\text{EnvyFree}[3, 3]$ allocations - envy-free and proportional for 3 agents with connected pieces.

```

One of :
  — Alice:Equalize(3)
  — Alice:Equalize(3); Bob:Equalize(2)
  — Alice:Equalize(3); Carl:Equalize(2)
  — Bob:Equalize(3)
  — Bob:Equalize(3); Alice:Equalize(2)
  — Bob:Equalize(3); Carl:Equalize(2)
  — Carl:Equalize(3)
  — Carl:Equalize(3); Alice:Equalize(2)
  — Carl:Equalize(3); Bob:Equalize(2)

```

The "One of" statement means that the algorithm should try each of the 9 execution branches on paper, and check whether it "succeeds" (i.e. leads to an envy-free and proportional division). Whenever an execution branch succeeds, the algorithm stops and implements the resulting allocation on the real cake. We now prove that at least one branches indeed succeeds.

Lemma 9. *For every preferences of the agents, there is at least one branch of Algorithm 2 in which the resulting allocation is $\text{EnvyFree}[3, 3]$.*

Proof. It is convenient to normalize the valuations such that each agent values the entire cake as 3. Hence, proportionality requires that each agent receives a value of at least 1.

Note that in each branch, the agent that does the initial $\text{Equalize}(3)$ always has at least one whole piece to choose, so he always feels no envy and has a value of at least 1. It remains to prove that the same is true for the other two agents. I.e, in at least one branch, there is an envy-free allocation and for every agent there is a piece worth at least 1.

Assume, for the sake of the proof, that each agent marks two points in the interval $[0, 1]$ that partition it to three intervals equal in his eyes. Denote the equal pieces of agent X by: $\hat{1}_X$, $\hat{2}_X$ and $\hat{3}_X$, such that the value of \hat{i}_X to agent X is exactly 1.

Assume w.l.o.g. that the order of the first lines is A-B-C. Hence: $\hat{1}_A \subseteq \hat{1}_B \subseteq \hat{1}_C$. There are $3! = 6$ options for the order of the second lines.⁴ We treat each of these cases in turn. Each case is illustrated by a picture; the vertical gray lines in the pictures are the cuts made by the agent who does the " $\text{Equalize}(3)$ " in the successful branch.

⁴We ignore the fortunate case in which two or more agents make a mark in the exact same spot. This case can be handled by assuming an arbitrary order between these agents.

5.1 C-B-A

$\hat{1}_C$			$\hat{2}_C$		$\hat{3}_C$	
A	B	C	C	B	A	

Ask Carl to Equalize(3) by cutting the cake at the points marked by "C" in the above picture (the vertical gray lines). Both Alice and Bob value two pieces - $\hat{1}_C$ and $\hat{3}_C$ - as at least 1. This can be easily seen in the picture. E.g, the fact that Alice's leftmost mark is inside $\hat{1}_C$ means that Alice values a subset of $\hat{1}_C$ as exactly 1, so she values $\hat{1}_C$ as at least 1. The same is true for piece $\hat{3}_C$ and for Bob.

Ask either Alice or Bob to Equalize(2). At most one piece is trimmed, so for each agent, at least one remaining piece has value at least 1. Moreover, both the cutter and Carl have two preferred pieces, so by Lemma 6 an envy-free allocation exists. So the branch "Carl:Equalize(3); Alice:Equalize(2)" succeeds (the branch "Carl:Equalize(3); Bob:Equalize(2)" also succeeds, but we only need one successful branch).

5.2 C-A-B

$\hat{1}_C$			$\hat{2}_C$		$\hat{3}_C$	
A	B	C	C	A	B	

The analysis of the case C-B-A applies as-is to this case.

5.3 A-B-C

$\hat{1}_B$		$\hat{2}_B$			$\hat{3}_B$	
A	B	C	A	B	C	

Ask Bob to Equalize(3). If Alice and Carl prefer different pieces, then we are done - an envy-free allocation exists with the current 3 pieces, so by the EFP lemma the proportionality is 1/3. The branch "Bob:Equalize(3)" succeeds.

Otherwise, Alice and Carl prefer the same piece. This piece must be $\hat{2}_B$, since $\hat{1}_B$ is worth less than 1 for Carl and $\hat{3}_B$ is worth less than 1 for Alice. This means that $\hat{2}_B$ is worth more than 1 for both Alice and Carl. Hence, each of them has two pieces worth at least 1: $\hat{1}_B$ and $\hat{2}_B$ for Alice, $\hat{2}_B$ and $\hat{3}_B$ for Carl. This is the same situation as in the case C-B-A. An Equalize(2) by either Alice or Carl guarantees an envy-free and proportional division. So the branch "Bob:Equalize(3); Alice:Equalize(2)" succeeds.

5.4 B-A-C

$\hat{1}_B$		$\hat{2}_B$		$\hat{3}_B$	
A	B	C	B	A	C

Ask Bob to Equalize(3). Alice values two pieces - $\hat{1}_B$ and $\hat{3}_B$ - as at least 1. Ask Alice to Equalize(2). Alice still has two pieces with a value of at least 1. As for Carl, there are two cases: If Alice trimmed $\hat{1}_B$, then $\hat{3}_B$ remains untouched; its value for Carl is more than 1. If Alice trimmed $\hat{3}_B$, then she must have trimmed at or to the left of the second A (since its value for her must be at least the value of $\hat{1}_B$, which is at least 1). Hence, the value of the trimmed piece for Carl is still at least 1. So the branch "Bob:Equalize(3); Alice:Equalize(2)" succeeds.

Alice		Bob		Carl	
A	B	C	B	A	C

5.5 A-C-B

$\hat{1}_C$			$\hat{2}_C$		$\hat{3}_C$
A	B	C	A	C	B

The previous case, A-B-C-B-A-C, is symmetric to A-B-C-A-C-B. This can be seen by renaming the agents from A-B-C to B-C-A and reversing the order of lines. The branch "Carl:Equalize(3); Bob:Equalize(2)" succeeds.

5.6 B-C-A

The last sub-case is handled according to Alice's preferences - whether she prefers $\hat{1}_B$ (which contains $\hat{1}_A$) or $\hat{3}_C$ (which contains $\hat{3}_A$). Note that Alice values both these pieces as at least 1.

If Alice prefers $\hat{1}_B$, then ask Bob to Equalize(3). Alice values two pieces as at least 1. Ask her to Equalize(2). For Carl there are two cases: If Alice trimmed $\hat{1}_B$, then $\hat{3}_B$ remains untouched; its value for Carl is more than 1. If Alice trimmed $\hat{3}_B$, then she must have trimmed it at or to the left of the C mark, since she values $\hat{1}_B$ more than $\hat{3}_C$. Hence, the value of the trimmed piece for Carl is still at least 1. The branch "Bob:Equalize(3); Alice:Equalize(2)" succeeds.

Alice		Bob		Carl	
A	B	C	B	C	A

Algorithm 3 Finding allocations which are both EnvyFreeVIP[4, 4] and EnvyFree[4, 7] with connected pieces.

```

Alice:Equalize(4)
├─ One of:
│   └─ Bob:Equalize(2); Carl:Equalize(2)
│       └─ Bob:Equalize(3); Carl:Equalize(2)
│           └─ Carl:Equalize(2); Bob:Equalize(2)
│               └─ Carl:Equalize(3); Bob:Equalize(2)

```

If Alice prefers $\hat{3}_C$, then ask Carl to Equalize(3). Alice still values two pieces as at least 1. Ask her to Equalize(2). For Bob there are two cases: If Alice trimmed $\hat{3}_C$, then $\hat{1}_C$ remains untouched; its value for Bob is more than 1. If Alice trimmed $\hat{1}_C$, then she must have trimmed it at or to the right of the B mark, since she values $\hat{3}_C$ more than $\hat{1}_B$. Hence, the value of the trimmed piece for Bob is still at least 1. The branch "Carl:Equalize(3); Alice:Equalize(2)" succeeds.

Bob			Carl		Alice
A	B	C	B	C	A

This completes the correctness proof of the 3-agents division algorithm and with it, the proof of our Theorem 2.⁵ \square

6 Connected pieces and 4 agents

Encouraged by the performance of the algorithm of Section 5, we would like to extend it to produce a connected envy-free and proportional allocation for n agents. Unfortunately, the number of different cases becomes prohibitively large even for $n = 4$ agents. The equal partition of each agent is made by 3 parallel marks, so if we name the agents according to their 1st mark, the number of options for the following two marks is $(4!)^2 = 576$, and in general $(n!)^{n-2}$. The algorithm for each specific case may be short, but writing down all the different cases takes too long to be practical.

Therefore we walk in a different direction and present Algorithm 3 for 4 agents, which we call Alice, Bob, Carl and Dana. The main guarantee of this algorithm is:

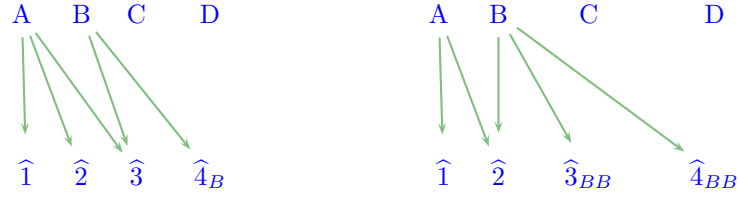
Lemma 10. *For every preferences of the agents, there is at least one branch of Algorithm 3 in which, in the resulting preference graph, each of Alice Bob and Carl prefers at least 2 pieces.*

⁵Note that the proof did not use all 9 branches in the algorithm. This is because the proof arbitrarily named the agents A, B and C according to the order of their leftmost division line. If the agents' names are given, each of the 9 branches may be required.

By Lemma 6, this implies that Algorithm 3 finds an envy-free allocation regardless of Dana’s preferences. Since at least one of Alice’s original pieces remains untouched, the resulting allocation is also $\text{EnvyFreeVIP}[4, 4]$ (Alice is the VIP). Since the total number of pieces in each branch is at most 7, the resulting allocation is $\text{EnvyFree}[4, 7]$, which proves our Theorem 3.

of Lemma 10. After $\text{Alice:Equalize}(4)$, there are four pieces on the table. We rename the pieces, if needed, such that Bob’s preference ordering on these pieces is: $\hat{1} \preceq \hat{2} \preceq \hat{3} \preceq \hat{4}$. Now there are $4! = 24$ possible preference orderings for Carl. Since checking 24 cases is a tedious task, we wrote a program in SageMath (Developers, 2015) to do it. Our program generates a textual proof that can be read and verified independently of the program itself (i.e, it is not required to believe that the program is bug-free in order to verify the proof). The entire proof is given in Appendix A. Below, we explain the typical cases in detail.

Mark piece \hat{i} after agent X cuts it during $\text{Equalize}(2)$ by \hat{i}_X and during $\text{Equalize}(3)$ by \hat{i}_{XX} . The following graphs show the preferences of Alice and Bob after Bob does $\text{Equalize}(2)$ (left) or $\text{Equalize}(3)$ (right):



Of the 24 possible preference orders of Carl, there are 6 cases in which the best piece of Carl is $\hat{1}$. This obviously remains Carl’s best piece after Bob trims some other pieces. This means that when Carl does $\text{Equalize}(2)$, piece $\hat{1}$ is trimmed. Looking at the left graph, we see that afterwards, each of Alice Bob and Carl prefers at least two pieces. This means that the branch starting with “Bob:Equalize(2)” succeeds. The same is true for the 6 cases in which Carl’s best piece is $\hat{2}$; we have already covered 12 out of 24 cases.

Next, consider the four cases in which Carl’s best piece is $\hat{3}$ and Carl’s second-best piece is $\hat{1}$ or $\hat{2}$. Then, after $\text{Bob:Equalize}(2)$, Carl’s best piece is $\hat{3}$ and trimming it leaves only one best piece for Bob, so the branch starting with “Bob:Equalize(2)” fails. On the other hand, after $\text{Carl:Equalize}(2)$, Bob’s best piece is still $\hat{4}$ and trimming it leaves two best pieces for Alice and Carl, so the branch starting with “Carl:Equalize(2)” succeeds. So far, 16 of 24 cases are covered.

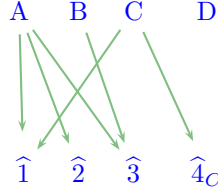
Next, consider the two cases in which Carl’s best piece is $\hat{4}$ and second-best piece is $\hat{1}$. After $\text{Bob:Equalize}(2)$, $\hat{4}_B$ may or may not be Carl’s best piece:



The easy case is that for Carl, $\hat{4}_B \preceq \hat{1} \preceq \hat{4}$ (left); then, Carl:Equalize(2) trims $\hat{1}$ and leaves two best pieces for Alice and Bob, and the branch starting with "Bob:Equalize(2)" succeeds. The hard case is $\hat{1} \preceq \hat{4}_B \preceq \hat{4}$ (right); then, Carl:Equalize(2) trims $\hat{4}_B$ and leaves only one best piece for Bob, and the branch starting with "Bob:Equalize(2)" fails. But now, consider the branch starting with "Carl:Equalize(2)". Carl trims piece $\hat{4}$ to make it equal to $\hat{1}$. But because $\hat{1} \preceq \hat{4}_B \preceq \hat{4}$, Carl must trim $\hat{4}$ to a *shorter* length than $\hat{4}_B$ (we assume that all agents trim the pieces from the same direction), so $\hat{4}_C \subseteq \hat{4}_B$. This means that, if the branch "Bob:Equalize(2)" fails, the following preference relation must be true globally (for *all* agents):

$$\hat{4}_C \preceq \hat{4}_B$$

In particular, it must be true for Bob. But by definition, for Bob, $\hat{4}_B$ and $\hat{3}$ are equal. Hence, for Bob: $\hat{4}_C \preceq \hat{3}$. This means that, after the initial Carl:Equalize(2), Bob's best piece is $\hat{3}$:



Now, Bob:Equalize(2) leaves two best pieces for both Alice and Carl, so the branch starting with "Carl:Equalize(2)" succeeds.

An almost identical argument applies in the two cases in which Carl's best piece is $\hat{4}$ and second-best piece is $\hat{2}$, so we already covered 20 of 24 cases; in each of these cases, either the branch starting with "B:Equalize(2)" or the branch starting with "C:Equalize(2)" succeeds.

The other two branches are used in the remaining four cases: in each of these cases, either the branch starting with "B:Equalize(3)" or the branch starting with "C:Equalize(3)" must succeed. The proof uses similar arguments to the one explained above: if the branch starting with "B:Equalize(3)" fails, then some global containment relations are implied between $\hat{4}_{BB}$ and $\hat{4}_{CC}$ and between $\hat{3}_{BB}$ and $\hat{3}_{CC}$. These relations imply that the branch starting with "C:Equalize(3)" must succeed. Although the proof is longer, the principle is the same so we leave the details to the printout in Appendix A. \square

Lemma 10 lets us improve the algorithm of Section 4. We note that the core of that algorithm is Lemma 8, which is exponential in nature - it requires that every known agent prefers $1 + 2^{u-1}$ pieces whenever there are u known agents. We would like to reduce this figure to $1 + u$. Since for $u \in \{1, 2\}$ these two expressions are equal, we focus on the case $u = 3$:

Lemma 11. *Suppose the n agents are divided to $n - 3$ agents whose preferences are known and 3 agent whose preferences are unknown. If every known agent prefers at least 4 pieces, then an envy-free allocation exists.*

Proof. Call the three unknown agents Bob, Carl and Dana. Apply Algorithm 3 without the first step "Alice:Equalize(4)". The assumption of the lemma implies that every known agent is in the same situation as Alice after the first step. This means that after the algorithm completes, all agents except the last unknown agent prefer two pieces. By Lemma 6, an envy-free allocation exists. \square

Lemma 11 can be plugged as a base case into Lemma 8 to get the following improved lemma:

Lemma 12. *Suppose the n agents are divided to $n - u$ agents whose preferences are known and u agents whose preferences are unknown, where $u \geq 3$. If every known agent prefers at least $1 + 3 \cdot 2^{u-3}$ pieces, then an envy-free allocation can be attained with a bounded number of queries.*

The proportionality of the division algorithm of Section 4 improves to $1/[\frac{3}{4} \cdot 2^{n-1} + 1]$.

Remark 2. *The symmetry of Algorithm 3 hints that it may be generalizable to 5 or more agents. In particular, we thought that an algorithm such as the following might work:*

```

Alice:Equalize(5)
One of:
├─ Bob:Equalize(2); Carl:Equalize(2); Dana:Equalize(2)
├─ Bob:Equalize(2); Carl:Equalize(3); Dana:Equalize(2)
├─ Bob:Equalize(3); Carl:Equalize(2); Dana:Equalize(2)
├─ Bob:Equalize(3); Carl:Equalize(3); Dana:Equalize(2)
├─ Bob:Equalize(4); Carl:Equalize(2); Dana:Equalize(2)
├─ Bob:Equalize(4); Carl:Equalize(3); Dana:Equalize(2)
└─ [and similarly for the other 5 permutations of Bob,
    Carl, Dana]
```

We checked this possibility using our SageMath program, but the posets did not become cyclic - we found a specific combination of preferences in which all these branches fail to produce a saturated matching.⁶ This means that new techniques may be needed to advance from 4 to 5 agents.

⁶One such case is when Bob's preference order is $1 \preceq 2 \preceq 3 \preceq 4 \preceq 5$, Carl's order is also $1 \preceq 2 \preceq 3 \preceq 4 \preceq 5$ while Dana's order is $1 \preceq 3 \preceq 2 \preceq 4 \preceq 5$. See <http://github.com/erelsgl/envy-free> for the source code and proof.

Algorithm 4 Finding $\text{EnvyFree}[4, 4]$ allocations with disconnected pieces.

Let $C' = C$.

For $i = 1$ to 4:

Rename A_i to "Alice";

Divide C' using Algorithm 3;

Let $C' =$ the subset that remained unallocated.

7 Disconnected pieces and 4 agents

In this and the following section, we use our results from the connected case to prove better improved proportionality bounds in the disconnected case. We show that, if the pieces may be disconnected, we can have an envy-free division in which the value of each agent is arbitrarily close to $1/n$, in bounded time. This is done using two general reduction lemmas which rely on envy-free-VIP algorithms.

Lemma 13. (*Weak Reduction Lemma*) For every n and $M \geq n$,

If there is an algorithm for finding $\text{EnvyFreeVIP}[n, M]$ allocations using $T(n)$ queries,

then there is an algorithm for finding $\text{EnvyFree}[n, M]$ allocations using $n \cdot T(n)$ queries.

Proof. (generalizing an idea of Saberi and Wang (2009)). The idea is to use the existing algorithm n times, each time on the remainder of the previous time and with a different agent as the VIP. This ensures that all agents enjoy the VIP proportion of $1/M$.

Let C be the original cake. Run $\text{EnvyFreeVIP}[n, M]$ on C with agent A_1 as the VIP. The result is an allocation of a certain subset of C (say, $C' \subseteq C$) with the following properties:

- The allocation of C' is envy-free.
- Hence, by the pigeonhole principle, every agent A_i has a value of at least $V_i(C')/n \geq V_i(C')/M$.
- Moreover, A_1 holds a value of at least $V_1(C)/M$.

If $C' = C$, then we are done since every agent A_i holds a value of at least $V_i(C)/M$. Otherwise, there is a remainder, $\overline{C'} = C \setminus C'$, that should be divided. Run $\text{EnvyFreeVIP}[n, M]$ on that remainder with A_2 as the VIP. The result is an allocation of a certain subset $C'' \subseteq \overline{C'}$ with the following properties:

- The allocation of C'' is envy-free.
- Hence, by the pigeonhole principle, every agent A_i holds a value of at least $V_i(C'')/n \geq V_i(C'')/M$.
- Moreover, A_2 has a value of at least $V_2(\overline{C'})/M$.

Algorithm 5 Finding $\text{EnvyFreeVIP}[n, \frac{n}{1-\epsilon}]$ allocations with disconnected pieces.

Rename the agents such that the VIP is A_1 .

Let $C' = C$.

For $t = 1$ to $\lceil (2^{n-2} + 1) \ln(1/\epsilon)/n \rceil$:

Divide C' using Algorithm 1;

Let $C' =$ the subset that remained unallocated.

Combining the two previous allocations, we now have an allocation of $C' \cup C''$, with the following properties:

- The allocation of $C' \cup C''$ is envy-free (since it is a combination of two envy-free allocations).
- Hence, by the pigeonhole principle, every A_i has a value of at least $V_i(C' \cup C'')/n \geq V_i(C' \cup C'')/M$.
- A_1 (still) has a value of at least $V_1(C)/M$, since nothing was taken from him.
- A_2 has a value of at least $V_2(C')/M + V_2(\overline{C'})/M$, which is at least $V_2(C)/M$.

So after the second division, we have an envy-free division in which both A_1 and A_2 hold at least $1/M$ of their total cake value.

If $C' \cup C'' = C$ then we are done. Otherwise, there is a remainder $\overline{C' \cup C''}$ that should be divided. Continue in the same way: run $\text{EnvyFreeVIP}[n, M]$ on that remainder with agent A_3 as the VIP, then with A_4 as the VIP, and so on. It is easy to prove by induction that, after at most n runs, all agents have at least $1/M$ of their total cake value. \square

The Weak Reduction Lemma is most useful in the case $M = n$. Note that an $\text{EnvyFree}[n, n]$ allocation is both envy-free *and proportional*. The Weak Reduction Lemma implies that such an allocation can be found using an algorithm which guarantees a value of at least $1/n$ to a *single* VIP agent.

The 4-agent algorithm of Section 6 indeed finds $\text{EnvyFreeVIP}[4, 4]$ allocations using a constant number of queries. This proves our Theorem 4.

8 Disconnected pieces and n agents

Currently we don't have an $\text{EnvyFreeVIP}[n, n]$ algorithm for $n \geq 5$, so we don't know if $\text{EnvyFree}[n, n]$ exists for $n \geq 5$. However, the following lemma allows us to approach it to any desired precision.

Lemma 14. (*Strong Reduction Lemma*) For every n , $M > n$ and $\epsilon > 0$:

If there is an algorithm for finding $\text{EnvyFreeVIP}[n, M]$ allocations using $T(n)$ queries,

then there is an algorithm for finding $\text{EnvyFreeVIP}[n, \frac{n}{1-\epsilon}]$ allocations using $\frac{M \ln(1/\epsilon)}{n} \cdot T(n)$ queries.

Proof. The main idea is to use the existing algorithm many times, each time on the remainder of the previous time, with the same agent as the VIP. The value of the VIP agent grows like a geometric series and converges to $1/n$. Hence, after a sufficient number of runs, the VIP agent's value is at least $(1 - \epsilon)/n$.

The proof uses the following notation:

- t - the number of times the $\text{EnvyFreeVIP}[n, M]$ algorithm has been run on successive remainders.
- C_t ($t \geq 0$) - the total cake that has been allocated after time t . Initially $C_0 = \emptyset$.
- C'_t ($t \geq 1$) - the cake actually allocated at time t (so $C_t = \cup_{j=1}^t C'_j$).
- V_t ($t \geq 0$) - the total value held by the VIP agent after time t . Initially $V_0 = 0$.
- V'_t ($t \geq 1$) - the value given to the VIP agent at time t (so $V_t = \sum_{j=1}^t V'_j$).

Lemma 15. *In every time $t \geq 1$:*

$$V'_t \geq [1 - nV_{t-1}]/M$$

Proof. Since all allocations are envy-free, the cumulative allocation of C_{t-1} is also envy-free. This means that the VIP agent, like all other agents, holds at least a proportional share of it:

$$V_{t-1} \geq V(C_{t-1})/n$$

Note that the cake that has to be divided at time t is $C \setminus C_{t-1}$. At time t , the VIP agent receives at least a fraction $1/M$ of it:

$$V'_t \geq [1 - V(C_{t-1})]/M$$

Combining the previous two inequalities gives the desired inequality. \square

Lemma 16. *In every time $t \geq 0$:*

$$V_t \geq \frac{1 - (1 - n/M)^t}{n}$$

Proof. By induction on t . For $t = 0$, by definition $V_0 = 0$. Suppose the claim is true for t , so there is a constant $d \geq 0$ such that:

$$V_t = \frac{1 - (1 - n/M)^t}{n} + d$$

By Lemma 15:

$$V'_{t+1} \geq [1 - nV_t]/M$$

By the induction assumption, $1 - nV_t = (1 - n/M)^t - nd$. Hence:

$$V'_{t+1} \geq [(1 - n/M)^t]/M - nd/M$$

So:

$$V_{t+1} = V_t + V'_{t+1} \geq \frac{1 - (1 - n/M)^t + (n/M)(1 - n/M)^t}{n} + d(1 - n/M)$$

Because $M \geq n$, the rightmost term is positive and we get the desired inequality:

$$V_{t+1} \geq \frac{1 - (1 - n/M)^{t+1}}{n}$$

□

By Lemma 16, to get a value of at least $V_t \geq (1 - \epsilon)/n$, it is sufficient to choose t such that:

$$(1 - n/M)^t \leq \epsilon$$

It is sufficient to take:

$$t = \frac{\ln \epsilon}{\ln(1 - n/M)} = \frac{\ln(1/\epsilon)}{-\ln(1 - n/M)}$$

By the log inequality: $\ln(1 - n/M) < -n/M$. Hence the required time is at most:

$$t \leq \frac{\ln(1/\epsilon)}{n/M} = \frac{M \ln(1/\epsilon)}{n}$$

□

By combining the two reduction lemmas we get:

Corollary 1. *For every n , $M > n$ and $\epsilon > 0$:*

If there is an algorithm for finding $\text{EnvyFreeVIP}[n, M]$ allocations using $T(n)$ queries,

then there is an algorithm for finding $\text{EnvyFree}[n, \frac{n}{1-\epsilon}]$ allocations using $M \ln(1/\epsilon) \cdot T(n)$ queries.

In the algorithm of Section 4, the first cutter cuts $2^{n-2} + 1$ equal pieces. Hence, the resulting allocations are $\text{EnvyFreeVIP}[n, 2^{n-2} + 1]$. The total number of queries is $O(2^n)$. Hence:

Corollary 2. *For every n and $\epsilon > 0$, there exists an algorithm for finding $\text{EnvyFree}[n, \frac{n}{1-\epsilon}]$ allocations using $O(4^n \ln(1/\epsilon))$ queries.*

This completes the proof of our Theorem 5.

9 Future Work

The two main questions left open by the present paper are:

- In the case of connected pieces: is there a bounded-time algorithm for finding envy-free and proportional allocations for 4 or more agents?
- In the case of disconnected pieces: is there an a bounded-time algorithm for finding envy-free-VIP allocations for 5 or more agents, in which the VIP agent receives at least $1/n$? Such an algorithm can be used as a building block both for finding envy-free and proportional allocations with free disposal (as in Section 7) and for finding envy-free allocations of an entire cake (as in Appendix B).

APPENDIX

A Automatically-generated proof for 4-agent envy-free-VIP algorithm

For convenience, we repeat here the algorithm of Section 6:

```
Alice:Equalize(4)
One of:
├─ Bob:Equalize(2); Carl:Equalize(2)
├─ Bob:Equalize(3); Carl:Equalize(2)
├─ Carl:Equalize(2); Bob:Equalize(2)
└─ Carl:Equalize(3); Bob:Equalize(2)
```

Below, when the proof says e.g. that "b:Equalize(2)... always succeeds", it means that the execution branch starting with "Bob:Equalize(2)" necessarily results in a preference graph in which Alice, Bob and Carl each prefer two pieces. Then, by Lemma 6 an envy-free allocation exists. Conversely, when the proof says that "this must fail", it means that the execution branch necessarily does not result in such a preference graph. The proof systematically checks all possible preference relations and proves that in all cases, at least one of the four execution branches must succeed. The source code of the program used to generate the proof is available in <https://github.com/erelsgl/envy-free>. We emphasize that the proof can be read and verified without the source code, so the correctness of the proof does not depend on the correctness of the code.

Initially, agent a cuts four equal pieces: 1,2,3,4 .

Assume w.l.o.g. that b's preferences are $1 < 2 < 3 < 4$.

Consider the following 24 cases regarding the preferences of c:

CASE 1 OF 24 : c's order is $4 < 3 < 2 < 1$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 2 OF 24 : c's order is $4 < 3 < 1 < 2$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 3 OF 24 : c's order is $4 < 2 < 3 < 1$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 4 OF 24 : c's order is $4 < 2 < 1 < 3$:
b:Equalize(2) makes b's best pieces: $3=4b$. This must fail because of c.
c:Equalize(2) makes c's best pieces: $1=3c$. This always succeeds.

CASE 5 OF 24 : c's order is $4 < 1 < 3 < 2$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 6 OF 24 : c's order is $4 < 1 < 2 < 3$:
b:Equalize(2) makes b's best pieces: $3=4b$. This must fail because of c.
c:Equalize(2) makes c's best pieces: $2=3c$. This always succeeds.

CASE 7 OF 24 : c's order is $3 < 4 < 2 < 1$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 8 OF 24 : c's order is $3<4<1<2$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 9 OF 24 : c's order is $3<2<4<1$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 10 OF 24 : c's order is $3<2<1<4$:
b:Equalize(2) makes b's best pieces: $3=4b$. This may fail in 1 case : c prefers 4b to 1 2 3 .
Assume the case c prefers 4b to 1 2 3. Then:
c:Equalize(2) makes c's best pieces: $1=4c$, so globally: $4c<4b$. This always succeeds.

CASE 11 OF 24 : c's order is $3<1<4<2$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 12 OF 24 : c's order is $3<1<2<4$:
b:Equalize(2) makes b's best pieces: $3=4b$. This may fail in 1 case : c prefers 4b to 1 2 3 .
Assume the case c prefers 4b to 1 2 3. Then:
c:Equalize(2) makes c's best pieces: $2=4c$, so globally: $4c<4b$. This always succeeds.

CASE 13 OF 24 : c's order is $2<4<3<1$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 14 OF 24 : c's order is $2<4<1<3$:
b:Equalize(2) makes b's best pieces: $3=4b$. This must fail because of c.
c:Equalize(2) makes c's best pieces: $1=3c$. This always succeeds.

CASE 15 OF 24 : c's order is $2<3<4<1$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 16 OF 24 : c's order is $2<3<1<4$:
b:Equalize(2) makes b's best pieces: $3=4b$. This may fail in 1 case : c prefers 4b to 1 2 3 .
Assume the case c prefers 4b to 1 2 3. Then:
c:Equalize(2) makes c's best pieces: $1=4c$, so globally: $4c<4b$. This always succeeds.

CASE 17 OF 24 : c's order is $2<1<4<3$:
b:Equalize(2) makes b's best pieces: $3=4b$. This must fail because of c.
c:Equalize(2) makes c's best pieces: $4=3c$. This must fail because of b.
b:Equalize(3) makes b's best pieces: $2=3bb=4bb$. This may fail in 1 case : c prefers 1 to 2 3bb 4bb .
Assume the case c prefers 1 to 2 3bb 4bb. Then:
c:Equalize(3) makes c's best pieces: $1=4cc=3cc$, so globally: $4bb<4cc$ 3bb<3cc . This always succeeds.

CASE 18 OF 24 : c's order is $2<1<3<4$:
b:Equalize(2) makes b's best pieces: $3=4b$. This may fail in 2 cases : c prefers 4b to 1 2 3; c prefers 3 to 1 2 4b .
Assume the case c prefers 4b to 1 2 3. Then:
c:Equalize(2) makes c's best pieces: $3=4c$, so globally: $4c<4b$. This may fail in 1 case : b prefers 3 to 2 1 4c .
Assume the case b prefers 3 to 2 1 4c. Then:
b:Equalize(3) makes b's best pieces: $2=3bb=4bb$. This may fail in 1 case : c prefers 1 to 2 3bb 4bb .
Assume the case c prefers 1 to 2 3bb 4bb. Then:
c:Equalize(3) makes c's best pieces: $1=3cc=4cc$, so globally: $3bb<3cc$ 4bb<4cc . This always succeeds.
Assume the case c prefers 3 to 1 2 4b. Then:
c:Equalize(2) makes c's best pieces: $3=4c$, so globally: $4b<4c$. This may fail in 1 case : b prefers 4c to 2 1 3 .
Assume the case b prefers 4c to 2 1 3. Then:
b:Equalize(3) makes b's best pieces: $2=3bb=4bb$. This may fail in 1 case : c prefers 1 to 2 3bb 4bb .
Assume the case c prefers 1 to 2 3bb 4bb. Then:
c:Equalize(3) makes c's best pieces: $1=3cc=4cc$, so globally: $3bb<3cc$ 4bb<4cc . This always succeeds.

CASE 19 OF 24 : c's order is $1<4<3<2$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 20 OF 24 : c's order is $1<4<2<3$:
b:Equalize(2) makes b's best pieces: $3=4b$. This must fail because of c.
c:Equalize(2) makes c's best pieces: $2=3c$. This always succeeds.

CASE 21 OF 24 : c's order is $1<3<4<2$:
b:Equalize(2) makes b's best pieces: $3=4b$. This always succeeds.

CASE 22 OF 24 : c's order is $1<3<2<4$:

b:Equalize(2) makes b's best pieces: $3=4b$. This may fail in 1 case : c prefers $4b$ to $1\ 2\ 3$.
 Assume the case c prefers $4b$ to $1\ 2\ 3$. Then:
 c:Equalize(2) makes c's best pieces: $2=4c$, so globally: $4c<4b$. This always succeeds.

CASE 23 OF 24 : c's order is $1<2<4<3$:
 b:Equalize(2) makes b's best pieces: $3=4b$. This must fail because of c.
 c:Equalize(2) makes c's best pieces: $4=3c$. This must fail because of b.
 b:Equalize(3) makes b's best pieces: $2=3bb=4bb$. This may fail in 1 case : c prefers 2 to $1\ 3bb\ 4bb$.
 Assume the case c prefers 2 to $1\ 3bb\ 4bb$. Then:
 c:Equalize(3) makes c's best pieces: $2=4cc=3cc$, so globally: $4bb<4cc\ 3bb<3cc$. This always succeeds.

CASE 24 OF 24 : c's order is $1<2<3<4$:
 b:Equalize(2) makes b's best pieces: $3=4b$. This may fail in 2 cases : c prefers $4b$ to $1\ 2\ 3$; c prefers 3 to $1\ 2\ 4b$.
 Assume the case c prefers $4b$ to $1\ 2\ 3$. Then:
 c:Equalize(2) makes c's best pieces: $3=4c$, so globally: $4c<4b$. This may fail in 1 case : b prefers 3 to $1\ 2\ 4c$.
 Assume the case b prefers 3 to $1\ 2\ 4c$. Then:
 b:Equalize(3) makes b's best pieces: $2=3bb=4bb$. This may fail in 1 case : c prefers 2 to $1\ 3bb\ 4bb$.
 Assume the case c prefers 2 to $1\ 3bb\ 4bb$. Then:
 c:Equalize(3) makes c's best pieces: $2=3cc=4cc$, so globally: $3bb<3cc\ 4bb<4cc$. This always succeeds.

Assume the case c prefers 3 to $1\ 2\ 4b$. Then:
 c:Equalize(2) makes c's best pieces: $3=4c$, so globally: $4b<4c$. This may fail in 1 case : b prefers $4c$ to $1\ 2\ 3$.
 Assume the case b prefers $4c$ to $1\ 2\ 3$. Then:
 b:Equalize(3) makes b's best pieces: $2=3bb=4bb$. This may fail in 1 case : c prefers 2 to $1\ 3bb\ 4bb$.
 Assume the case c prefers 2 to $1\ 3bb\ 4bb$. Then:
 c:Equalize(3) makes c's best pieces: $2=3cc=4cc$, so globally: $3bb<3cc\ 4bb<4cc$. This always succeeds.

Q.E.D!

B Envy-free division of an entire cake

Recently, Aziz and Mackenzie (2015) have made an important breakthrough in the search for bounded-time envy-free cake-cutting algorithms. They presented the first bounded-time algorithm for envy-free cake-cutting of an *entire cake* to 4 agents. In this appendix, we use our envy-free-VIP algorithm of Section 6 to present their results in a simpler and more general way.

B.1 The domination graph

The main new concept required for envy-free division of an entire cake is **domination**.⁷ We say that Alice dominates Bob if Alice won't envy Bob even if the entire remaining cake is given to Bob.

To see how a domination relation is created, consider again the envy-free-VIP algorithm for three agents, presented in Subsection 3.2. Alice does Equalize(3) and Bob does Equalize(2), cutting his favorite piece, say $\hat{3}$, to make it equal to his second-best, say $\hat{2}$. Mark the trimmed piece $\hat{3}_B$ and the trimmings $\hat{4}$, so that $\hat{3} = \hat{3}_B \cup \hat{4}$. Suppose Carl's best piece is $\hat{2}$. So Carl takes $\hat{2}$, Bob takes $\hat{3}_B$, Alice takes $\hat{1}$, and $\hat{4}$ remains for the next round. Now, the following equalities hold for Alice:

$$V_A(\hat{1}) = V_A(\hat{2}) = V_A(\hat{3}) = V_A(\hat{3}_B) + V_A(\hat{4})$$

⁷This concept originated with Brams and Taylor (1996), who used the term "irrevocable advantage". Aziz and Mackenzie introduced the shorter term "domination".

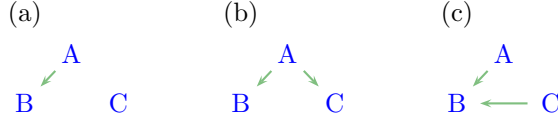


Figure 1: Domination graphs with 3 agents.

Hence, even if the entire remainder $(\hat{4})$ is given to Bob, Alice will not envy. This means that Alice dominates Bob.

The domination relation can be described by a **domination graph**. In a domination graph, the nodes are the agents and an edge between two agents means that the source node dominates the target node.

Three domination graphs are shown in Figure 1. The graph (a) is generated after a single run of the algorithm, in which Alice dominates Bob (as explained above). Graph (b) can possibly occur after a second run of the same algorithm, if in the second run Carl takes the trimmed piece so Alice dominates Carl too. Graph (c) can occur after a second run of the algorithm in which Carl is the cutter, if Bob takes the trimmed piece.

In general, when an envy-free algorithm is repeatedly executed, each time on the remainder of the previous time, edges are added to the domination graph but never removed.

B.2 Solvable domination graphs

A convenient approach to the envy-free division problem is to reduce a given instance to a simpler instance that we already know how to solve. Formally:

Definition 1. A domination graph of an envy-free cake-cutting problem for n agents is called **solvable** if, once the state of the division arrives at that domination graph, the problem can be reduced to one or more envy-free cake-cutting problems for less than n agents.

The domination graphs in Figure 1 (b) and (c) are solvable: in (b), the problem can be reduced to a 2-agent division between Bob and Carl, since Alice dominates both of them; in (c), the entire remainder can be given to Bob, since he is dominated by both Alice and Carl. In general:

Lemma 17. If, in a domination graph for n agents, there is a partition of the agents to two nonempty groups such that every agent in group #2 dominates all agents in group #1, then the domination graph is solvable.

Proof. An envy-free division of the entire cake can be found by letting the agents in group #1 (whose number is less than n) divide the remainder among them in an envy-free way. \square

Figure 2 shows three graphs for 4 agents that are solvable by Lemma 17.

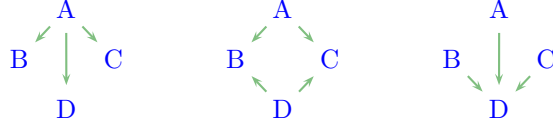


Figure 2: Domination graphs with 4 agents. All are solvable by Lemma 17.

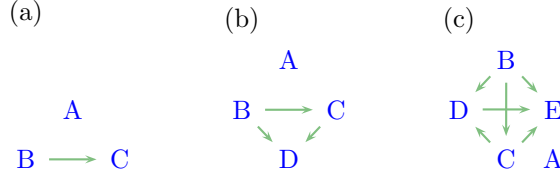


Figure 3: Domination graphs solvable by Lemma 18.

Another kind of solvable domination graphs is described in the following lemma.

Lemma 18. *If there is a sequence of $n - 1$ agents, A_2, \dots, A_n , such that every agent dominates the following agents (every agent A_i dominates every agent A_j for all $2 \leq i < j$), then the domination graph is solvable.*

Proof. The remaining cake can be divided in the following way: A_1 (the agent not in the sequence) cuts the cake to n equal parts. Then, the agents take pieces in the order A_n, \dots, A_2, A_1 . The agents in the sequence are not envious, because every agent dominates the agents that took pieces before him, and prefers his piece to the pieces taken by agents after him. A_1 is also not envious because all pieces are equal in his eyes. \square

Figure 3 shows graphs for 3, 4 and 5 agents, that are solvable by Lemma 18. In (a), the sequence is $\{B, C\}$, and the solution of Lemma 18 yields the well-known Selfridge-Conway algorithm. In (b) the sequence is $\{B, C, D\}$ and in (c) it is $\{B, C, D, E\}$.

Combining the two previous lemmas gives a stronger lemma:

Lemma 19. *If there is a set of $n - 1$ agents, each of whom dominates $n - 2$ agents, then the domination graph is solvable.*

Proof. Suppose that each of the agents A_2, \dots, A_n dominates $n - 2$ agents. Consider the following two cases:

Case #1: all agents A_2, \dots, A_n dominate A_1 . Then by Lemma 17 the domination graph is solvable.

Case #2: there is an agent in A_2, \dots, A_n , say A_2 , that does not dominate A_1 . Hence, this agent must dominate all the other agents A_3, \dots, A_n .

Regarding this smaller set of agents, there are again two cases:

Case #2.1: all agents A_3, \dots, A_n dominate both A_1 and A_2 . Then by Lemma 17 the domination graph is solvable.

Case #2.2: there is an agent in A_3, \dots, A_n , say A_3 , that does not dominate A_1 or does not dominate A_2 . Hence, this agent must dominate all the other agents A_4, \dots, A_n .

Regarding this smaller set of agents, we can continue the same line of reasoning. Finally we conclude that, either the domination graph is solvable by Lemma 17, or there exists a sequence of agents (A_2, \dots, A_n) such that each agent dominates the following agents in that sequence; then by Lemma 18 the domination graph is solvable. \square

Lemma 19 implies that the problem of envy-free division of an entire cake among n agents can be reduced to the following problem:

Find an envy-free allocation of a part of a cake, such that a pre-specified VIP agent dominates $n - 2$ agents.

We now show that this reduced problem can be solved for $n = 4$ agents.

B.3 The *Equalize** query

First, we want to guarantee that after *every* run of an envy-free-VIP algorithm, the VIP agent (the cutter) will dominate one of the agents. In order to guarantee this, we must change the semantics of the Equalize query. We call the changed query *Equalize**. An *Equalize**(k) query asks an agent to cut his best $k - 1$ pieces, such that the trimmed pieces will be equivalent to the agent's k -th best piece. For example, an *Equalize**(2) query to Bob in the above example implies the following question: "where would you cut piece $\hat{3}$, your currently favorite piece, such that the trimmed piece will be equivalent to $\hat{2}$?" Note that in this case (in contrast to the Equalize query), the trimmings may be more valuable than the trimmed piece. When *Equalize** queries are used, the agents are not allowed to choose the trimmings; the trimmings are kept for later iterations. The agents are only allowed to take the trimmed pieces (hence, in contrast to the algorithms using Equalize, there is no guarantee on the proportionality of the allocation after a single run of the algorithm). Since the number of original pieces is n , all trimmed pieces must be taken.

Based on the above observation, we now generalize a lemma proved by Aziz and Mackenzie (2015) from 4 to n agents.

B.4 Creating a single domination-edge from the VIP

Lemma 20. *Let C be a cake and X an envy-free division of a subset $C' \subset C$ among n agents. Denote the remaining cake by $\overline{C'} = C \setminus C'$. Suppose that for two agents (e.g. Alice and Bob) the following holds:*

$$V_A(X_A) - V_A(X_B) \geq V_A(\overline{C'})/k$$

where $k < n$. Then, after running an envy-free-VIP algorithm a bounded number $f(n)$ times with Alice as the VIP, Alice will dominate Bob.

Proof. Each run of an envy-free-VIP algorithm gives the VIP (Alice) a value of at least $1/n$. Hence, the value of the remaining cake decreases by a factor of at least $(n-1)/n$. Let $f(n) = \frac{\log n}{\log(n) - \log(n-1)}$. Note that $f(n) > \frac{\log k}{\log(n/(n-1))}$. Hence, after $f(n)$ iterations, the value of the remaining cake for Alice is at most $V_A(\overline{C'})/k$. When this happens, the difference between Alice's value to Bob's value (in Alice's eyes) is more than the value of the remainder; hence Alice dominates Bob. \square

Motivated by this lemma, we say that Alice k -dominates Bob, if $V_A(X_A) - V_A(X_B) \geq V_A(\overline{C'})/k$. If Alice k -dominates Bob (where $k < n$), then after a number of steps which is a bounded function of n , Alice will dominate Bob. Hence, from now on, we add an edge in the domination graph whenever the source node k -dominates the target node for some $k < n$.

Lemma 21. *After a run of an envy-free-VIP algorithm for n agents, the VIP k -dominates at least one other agent, where $k < n$.*

Proof. An envy-free-VIP algorithm starts by the VIP agent (say, Alice) cutting the cake to n equal pieces. Then, a certain number $k < n$ of pieces are trimmed. Consider the following two cases.

- (a) $k = 0$: all n pieces are taken with no trimmings. Then, the division is fully envy-free and no cake is left, so domination is trivial.
- (b) $1 \leq k < n$: the divided cake is $C' \subseteq C$, and the remainder is $\overline{C'} = C \setminus C'$. This remainder is the union of the k trimmings. Mark by \hat{i}_X the trimming taken from piece \hat{i} . Then:

$$\overline{C'} = \cup_{i=1}^k \hat{i}_X$$

By the additivity of Alice's value measure:

$$V_A(\overline{C'}) = \sum_{i=1}^k V_A(\hat{i}_X)$$

Assume, without loss of generality, that the trimming of piece $\hat{1}$ has the largest value for Alice (Aziz and Mackenzie call such piece the **significant piece**). Then, by the pigeonhole principle, its value for Alice is at least $1/k$ the value of the remaining cake, so:

$$V_A(\hat{1}_X) \geq V_A(\overline{C'})/k$$

This means that Alice k -dominates the agent that took piece $\hat{1}$. \square

Lemma 21 guarantees that, after each run of an envy-free-VIP algorithm, the domination graph contains an edge going from the VIP agent to another agent. Hence, n domination edges can be created by running the algorithm n

times with different VIP agents. But this may be insufficient to attain a solvable domination graph. The worst case is that, whenever a certain agent is the VIP, the same other agent takes the significant piece and hence the same domination edge is added again and again. Fortunately, Aziz and Mackenzie (2015) found a way to shift domination edges to other agents.

B.5 Creating two domination-edge from the VIP

Suppose there are n agents and Alice is the VIP. Suppose that after the first run, Alice dominates Bob. Our goal now is to make Alice dominate another agent. We run the algorithm again, this time keeping Bob as the last agent (the agent that does not trim). The other $n - 2$ agents trim some of the pieces, until each of the first $n - 1$ agents prefers at least two pieces (see Section 3.2 for a description on how it is done when $n = 3$ and Section 6 for the case $n = 4$). Now, Bob has to choose a piece. Suppose w.l.o.g. that Bob's best piece is $\hat{1}$ and his second-best piece is $\hat{2}$. There are two cases:

Easy case: $\hat{1}$ is not the significant piece. Then, another agent takes the significant piece and is k -dominated by Alice. Now two different domination edges emanate from Alice, as we wanted.

Hard case: $\hat{1}$ is the significant piece. Consider now what happens if Bob takes $\hat{2}$ instead of $\hat{1}$. The other agents will not care, since each of the other agents prefers at least two pieces. But then Bob might envy the agent (say, Carl) who takes $\hat{1}$. In this case, we say that Bob *competes* with Carl on the significant piece. Let $\Delta V = V_B(\hat{1}) - V_B(\hat{2})$. If Bob takes $\hat{1}$ then Bob has an advantage of at least ΔV over Carl; if Bob takes $\hat{2}$ then Bob has an envy of ΔV at Carl (Carl does not envy Bob in either case since Carl prefers two pieces).

Now, suppose the algorithm is run again and again with the same VIP, and each time we fall into the same hard case in which the same Bob prefers the significant piece. Eventually (after at most n runs), Bob competes with an agent with whom he already competed in the past (e.g, Bob competes Carl again). Now, we ask Bob in which of these two runs the ΔV is larger. If the ΔV was larger in the first run, then in the second run we give Bob his second-best piece; if the ΔV was larger in the second run, then in the first run we change the allocation and give Bob his second-best piece. In either case, Bob will not be envious since the larger ΔV cancels the envy caused by the smaller ΔV .

The above discussion can be summarized in the following lemma:

Lemma 22. *After at most n runs of an envy-free-VIP algorithm for n agents, the VIP k -dominates at least two other agents.*

Plugging Lemma 22 into Lemma 19 yields Aziz & Mackenzie's envy-free cake-cutting algorithm for 4 agents.

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